Dynamics of relativistic magnetized blast waves

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The dynamics of a relativistic blast wave propagating through magnetized medium is considered taking into account possible inhomogeneities of density and magnetic field and additional energy supply. Under the simplifying assumption of a spherically symmetric explosion in a medium with toroidal magnetic field self-similar solutions for the internal dynamics of the flow are derived. In the weakly magnetized case, when the bulk of the flow may be described by the unmagnetized solutions, there is a strongly magnetized sheath near the contact discontinuity (when it exists). Self-similar solutions inside the sheath are investigated. In the opposite limit of strongly magnetized upstream plasma new analytical self-similar solutions are found. Possible application to the physics of Gamma-Ray Bursts is discussed.

1. Introduction

Recent observational advances in the area of Gamma-Ray Bursts (GRBs) (see, e.g., Ref 1 for a review) have stimulated research on the physics of strongly relativistic explosions. Most analytical results for the dynamics of strong explosions belong to the class of self-similar solutions. This approach allows great simplification in reducing a system of partial differential equations to the ordinary ones and often represents an asymptotic behavior of the flow. Following the seminal works of Sedov[2] (for non-relativistic strong explosions) and of Blandford & McKee[3] (B&M hereafter) a series of generalizing works have been done. Non-relativistic shock waves propagating in a magnetized medium have been considered[4–6] as well as generalizations of the B&M solutions for a wide class of density profiles[7,8]. Yet, until now no account of the dynamics of the magnetic field in relativistic blast waves has been made. A lack of the relativistic treatment of the blast wave in magnetized medium is noticeable, especially in view of heuristic interpretation of GRB emission as due to synchrotron emission in strongly magnetized relativistic blast waves (e.g., Ref 9). This is done in the present work. In a follow-up work[10] we will explore the internal structure of the relativisticly expanding magnetic cavity.

2. Formulation of the problem

We seek self-similar solutions to the relativistic dynamics of a spherical expansion of gas into a magnetized medium due to strong explosion. We assume that (i) time dependent energy source is located in the center; (ii) the magnetic field is toroidal and spherically symmetric; (ii) there is a cold spherically symmetric external medium with density n_1 (which may depend on radius).

The assumption of spherical symmetry with toroidal magnetic field is a controversial, but a commonly employed simplification. A number of works applied spherical approximation to MHD winds from pulsars[11–13] and to explosions in the Solar wind[5,6]. Other astrophysical setting where such situation may arise is an explosion in a preexisting cavity blown out by spherical magnetized stellar wind. Formally, our approach also applies to the equatorial region of an explosion in a constant magnetic field, where the shock velocity is perpendicular to the magnetic field.

The formal treatment of the problem, in which we rely mainly on the approach of B&M, starts with the set of relativistic magnetohydrodynamic equations which can be written in terms of conservation laws[15]

$$T^{ij}_{,i} = 0, \tag{1}$$

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$$F^{*\,ij}_{,i} = 0, \tag{2}$$
$$(\rho u^i)_{,i} = 0 \tag{3}$$

where

$$T^{ij} = (w + b^2) u^i u^j + (p + \frac{1}{2} b^2) g^{ij} - b^i \, b^j \tag{4}$$

is the energy-momentum tensor, $w = 4p$ is plasma proper enthalpy (upfront in the cold plasma $w_1 = n_1 m_i c^2$, where n_1 is external density, m_i is the ion mass, while behind the shock $w = 4p$ appropriate for relativistic plasma), $b^2 = b_i b^j$ is the plasma proper magnetic energy density times 4π, p is pressure, $u^i = (\gamma, \gamma\beta)$ are the plasma four-velocity, Lorentz-factor and three-velocity, g^{ij} is the metric tensor, $b_i = \frac{1}{2}\eta_{ijkl} u^j F^{kl}$ are the four-vector of magnetic field, Levy-Chevita tensor and electro-magnetic field tensor. It has also been implicitly assumed in the derivation of these equations that one of the electromagnetic invariants is not equal to 0 and the electro-magnetic stress energy tensor can be diagonalized (equivalently, this implies that there is a reference frame where the magnetic (or electric) field is not equal to 0).

Writing out Eqns (1-3) in coordinate form and assuming a spherically symmetric outflow with toroidal magnetic field, the conservation of energy and momentum (1), induction equation (2) and mass conservation give

$$\partial_t \left[(w + b^2)\gamma^2 - (p + b^2/2) \right] + \frac{1}{r^2} \partial_r \left[r^2 (w + b^2) \beta \gamma^2 \right] = 0 \tag{5}$$
$$\partial_t \left[(w + b^2)\gamma^2 \beta \right] + \frac{1}{r^2} \partial_r \left[r^2 \left((w + b^2)\beta^2\gamma^2 + (p + b^2/2) \right) \right] - \frac{2p}{r} = 0 \tag{6}$$
$$\partial_t \left[b\gamma \right] + \frac{1}{r} \partial_r \left[r b \beta \gamma \right] = 0 \tag{7}$$
$$\partial_t \left[\rho\gamma \right] + \frac{1}{r^2} \partial_r \left[r^2 \rho \beta \gamma \right] = 0 \tag{8}$$

Equation (5-8) should be complemented by the boundary conditions on the shock front. The jump conditions on the relativistic transverse magnetized shocks may be written as[12]

$$\begin{aligned}
\gamma^2 &= \frac{\gamma' - u'}{\gamma' + u'} \Gamma^2 = \Gamma^2 \times \left\{ \begin{array}{ll} \frac{1}{2} & \text{if } \sigma \ll 1 \\ \frac{1}{4\sigma} & \text{if } \sigma \gg 1 \end{array} \right. \\
p_2 &= \frac{m \, n_1 \, c^2}{4 \, u' \, \gamma'} \left(1 + \sigma \left(1 - \frac{\gamma_2{}'}{u_2{}'} \right) \right) \Gamma^2 = n_1 m c^2 \Gamma^2 \times \left\{ \begin{array}{ll} \frac{2}{3} & \text{if } \sigma \ll 1 \\ \frac{1}{8\sigma} & \text{if } \sigma \gg 1 \end{array} \right. \\
n_2 &= \frac{n_1 \Gamma}{u'} = n_1 \Gamma \times \left\{ \begin{array}{ll} 2\sqrt{2} & \text{if } \sigma \ll 1 \\ \frac{1}{\sqrt{\sigma}} & \text{if } \sigma \gg 1 \end{array} \right. \\
b_2 &= \frac{\Gamma}{u'} b_1 = \frac{\Gamma}{u'} \sqrt{\sigma n_1 m c^2} = \sqrt{n_1 m c^2} \Gamma \times \left\{ \begin{array}{ll} 2\,\sqrt{2\,\sigma} & \text{if } \sigma \ll 1 \\ 1 & \text{if } \sigma \gg 1 \end{array} \right.
\end{aligned} \tag{9}$$

where

$$u'^2 = \frac{1 + 10\,\sigma + 8\,\sigma^2}{16\,(1+\sigma)} + \frac{\sqrt{1 + 20\,\sigma\,(1+\sigma) + 64\,\sigma^2\,(1+\sigma)^2}}{16\,(1+\sigma)}, \qquad \gamma'^2 = u'^2 + 1 \tag{10}$$

are the four speed and Lorentz factor of the shocked fluid in the frame of the shock and

$$\sigma = \frac{b_1{}^2}{m \, c^2 \, n_1} \tag{11}$$

is the magnetization parameter in front of the shock. The subscripts denote the quantities in the unshocked (1) and shocked (2) media.

Relations (9) allow for the following parameterization

$$
\begin{aligned}
&p_2 = \Gamma^2 n_1 m c^2 f(\chi) \\
&\gamma^2 = \Gamma^2 g(\chi) \\
&n_2 = \Gamma n_1 n(\chi) \\
&b_2 = \Gamma b_1 h(\chi)
\end{aligned} \tag{12}
$$

with the boundary conditions $f(1) = f_0$, $g(1) = g_0$,$h(1) = h_0$, $n(1) = n_0$ (see below). [1]

Following B&M we choose the self-similar variable

$$
\chi = 1 + 2(m+1)\xi = [1 + 2(m+1)\Gamma^2](1 - r/t) \tag{13}
$$

where $\xi = (1 - r/R)\Gamma^2$, Γ is the Lorentz factor of the shock, $R = t\left(1 - 1/(2(m+1)\Gamma^2)\right)$ is the radius of the contact discontinuity and we assume the Lorentz factor scales with radius as $\Gamma^2 \propto t^{-m}$. We limit ourselves to the strongly relativistic case expanding all relations to first order in $1/\Gamma^2$.

Treating (χ, y), where $y = \Gamma^2$, as new independent variables we find

$$
\begin{aligned}
&\partial_t = -my\partial_y + ((m+1)(2y - \chi) + 1)\partial_\chi \\
&\partial_r = -(1 + 2(m+1)y)\partial_\chi \\
&\beta = 1 - \frac{1}{2yg} \\
&r = t\left(1 - \frac{\chi}{1 + 2(m+1)y}\right)
\end{aligned} \tag{14}
$$

The equations for the self-similar variables f, g, n and h read

$$
\begin{aligned}
&\frac{\mathcal{A}}{g}\frac{\partial \ln f}{\partial \chi} = -2\,\left(4(1-m) + (m-4)\,\chi g\right)\,f + \frac{2(1-m) + 3\,(m-2)\,\chi g}{(\chi g - 1)}\,h^2 \\
&\frac{\mathcal{A}}{g}\frac{\partial \ln g}{\partial \chi} = -\left(4 - 7\,m + 2\,(2+m)\,\chi g\right)\,f + 3\,(m-1)\,h^2 \\
&\frac{\mathcal{A}}{g}\frac{\partial \ln h}{\partial \chi} = -\frac{\left(2\,(m-4)\,\chi^2 g^2 + (8 - 11\,m)\,(\chi g - 1)\right)}{2\,(\chi g - 1)}\,f + \frac{3\,(m-1)}{2}\,h^2 \\
&\frac{\mathcal{A}}{g}\frac{\partial \ln n}{\partial \chi} = \frac{1}{2(\chi g - 1)}\left(-f\,\left(-12 + 11\,m + g\,(24 - 11\,m)\,\chi + 2\,g^2\,(m-6)\,\chi^2\right) + \right. \\
&\left. 3\,h^2\,\left(1 - m + g\,(m-3)\,\chi\right)\right) \\
&\mathcal{A} = (1+m)\,\left(2\,f\,\left(1 - 4\,\chi g + g^2\,\chi^2\right) - 3\,\chi g\,h^2\right)
\end{aligned} \tag{15}
$$

System (15) is a relativistic generalization of Eqns. (3.10-3.13) of Ref. 6 In the limit $h \to 0$ these relations reproduce the unmagnetized case of B&M. [2]

Sometimes it is more convenient to use use the functions $\tilde{g}, \tilde{f}, \tilde{n}, \tilde{h}$ and $\tilde{x} = \tilde{g}\chi$ with boundary conditions on the shock front $\tilde{g} = \tilde{f} = \tilde{n} = \tilde{h} = \tilde{x} = 1$:

$$
\begin{aligned}
&g = g_0\tilde{g}, \quad f = f_0\tilde{f}, \quad n = n_0\tilde{n}, \quad h = n_0\tilde{h}, \quad x \equiv g\chi = g_0\tilde{x} \\
&g_0 = \frac{u' + \gamma'}{\gamma' - u'} \approx \begin{cases} \frac{1}{2} & \sigma \ll 1 \\ \frac{1}{4\sigma} & \sigma \gg 1 \end{cases}
\end{aligned}
$$

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[1] We choose to work consistently with proper quantities, i.e. measured in the plasma rest frame. One should be careful in comparing our equation with B&M and Ref. 12

[2] The system (15) may in fact be transformed into a system of one ODE and three equations resolvable in quadratures if we change to a new coordinate $x = \chi\, g$ and introduce a magnetization parameter $\beta = f/h^2$.

$$
\begin{aligned}
f_0 &= \frac{1}{4\, u_2\, {\gamma_2}'}\left(1+\sigma\left(1-\frac{{\gamma_2}'}{{u_2}'}\right)\right) = \begin{cases} \frac{2}{3} & \sigma \ll 1 \\ \frac{1}{8\sigma} & \sigma \gg 1 \end{cases} \\
n_0 &= \frac{1}{u'} \approx \begin{cases} 2\sqrt{2} & \sigma \ll 1 \\ \frac{1}{\sqrt{\sigma}} & \sigma \gg 1 \end{cases} \\
h_0 &= \frac{\sqrt{\sigma}}{u'} \approx \begin{cases} 2\sqrt{2\sigma} & \sigma \ll 1 \\ 1 & \sigma \gg 1 \end{cases}
\end{aligned}
\tag{16}
$$

We can solve the above equations by direct numerical integration. (see Fig. 1 and 2). Simple analytical relations, though, can be obtained in the weakly, $\sigma \ll 1$, and strongly magnetized, $\sigma \gg 1$, limits.

3. Small σ limit

Relations for $\tilde{g}, \tilde{f}$ and $\tilde{n}$ in this case are the same as for the unmagnetized case (B&M). For the evolution of the overall passive magnetic field we find

$$
\frac{1}{\tilde{g}}\frac{\partial \ln \tilde{h}}{\partial \chi} = -\frac{\left(-16+22\,m+(8-11\,m)\;\chi\tilde{g}+(m-4)\;\chi^2\,\tilde{g}^2\right)}{2\;(1+m)\;(\chi\,\tilde{g}-2)\;(4-8\,\chi\,\tilde{g}+\chi^2\,\tilde{g}^2)}
\tag{17}
$$

which can be integrated using the known solution for $\tilde{g}$. In particular, for $m=3$, this can be resolved for $\tilde{h}(\chi)$:

$$
\tilde{h}(\chi) = \frac{1}{\chi}
\tag{18}
$$

Recall that in this case $\tilde{g}=1/\chi$, $\tilde{f}=1/\chi^{17/12}$ and $\tilde{n}=1/\chi^{5/4}$. The magnetization parameter, $\tilde{\beta}=\tilde{f}/\tilde{h}^2=1/\chi^{5/12}$ in this case is a decreasing function of χ, so if it is small on the shock it will always remain small.

A. Magnetosheath

When the energy contained inside the shock increases with time a power source is necessary. This can be an expanding piston or another fluid with a larger Lorentz factor. In both cases a contact discontinuity (CD) forms separating the external and internal fluids. As discussed by B&M the point $\chi\, g \equiv x = 1$ is the location of the contact discontinuity. As Eq. (17) (which is the small σ limit of exact relations) suggests, the magnetic field increases infinitely near the CD. In reality, of course, it will grow until approximate equipartition is reached forming a thin boundary layer. This is consistent with the non-relativistic consideration[16,5] where it was noted that however small the magnetic field is in the unshocked medium near the CD the magnetic field grows in magnitude and starts to dominate the dynamics of the thin boundary layer. In this section we study the dynamics of the magnetosheath.

To estimate the thickness of the magnetosheath we use the system (A2) using independent variable x ($k=l=0$ in this section). In the small σ limit the local magnetization parameter $\beta = h^2/f$ is

$$
\beta \sim (x-1)^{2(m-4)/(12-m)}
\tag{19}
$$

Since near the CD $g \sim$ constant the width of the magnetic sheath in coordinates x is

$$
\Delta x \propto \sigma^{(12-m)/2(4-m)}
\tag{20}
$$

The constant of proportionality here is a complicated function of m. The width of the magnetic sheath in coordinates χ is

$$
\Delta \chi = \frac{\Delta x}{g'_{CD}/g_{CD} + g_{CD}}
\tag{21}
$$

where g'_{CD} and g_{CD} are the values taken at the CD.

Next we study the structure of the boundary layer. Using x as independent variable and retaining leading terms near $x=1$ we find

$$
\begin{aligned}
\frac{\partial \ln f}{\partial x} &= -\frac{h^2\,(m-4)}{(6\,h^2 - f\,(m-12))\,(x-1)} \\
\frac{\partial \ln g}{\partial x} &= \frac{3\,h^2\,(m-1) + f\,(-8+5\,m)}{-6\,h^2 + f\,(m-12)} \\
\frac{\partial \ln h}{\partial x} &= \frac{f\,(m-4)}{(6\,h^2 - f\,(m-12))\,(x-1)} \\
\frac{\partial \ln n}{\partial x} &= \frac{\left(2\,f\,m + 3\,h^2\,(-1+3\,x)\right)}{2\,(6\,h^2 - f\,(m-12))\,(x-1)}
\end{aligned}
\tag{22}
$$

which immediately gives

$$
f + h^2/2 = C_1 \tag{23}
$$

where C_1 determines the total energy flux through the magnetosheath. For simplicity below we put C_1 equal to 1.

Using (23) we can integrate the equation for the magnetic field

$$
h^{m-12}\left(h^2-2\right)^6 (x-1)^{m-4} = C_2 \tag{24}
$$

where C_2 is a constant of integration. Relation (24) implicitly determines the self-similar structure of the magnetic field inside the boundary layer.

As we approach the CD, $x \to 1$,

$$
h \sim \sqrt{2} - (x-1)^{(4-m)/6} \tag{25}
$$

This shows that for $m < 4$, as $x \to 1$ the magnetic field goes to a constant on the CD: $h \to \sqrt{2}$. Similarly we find that $f \sim (x-1)^{(4-m)/6} \to 0$ and $n \sim \sqrt{x-1} \to 0$. Thus, pressure and density vanish at the CD while the magnetic field is finite. This result confirms that no matter how small the external magnetic field is, there will be a region near the CD where magnetic field pressure is dominant. This is qualitatively different from the hydromagnetic case where (for $m > 0$) the density was vanishing on the CD, resulting in very high temperatures (c.f. Ref 5).

The assumption of ideal MHD should break down somewhere inside the boundary layer where diffusive and dissipative processes are likely to play a very important role.

The solution inside the boundary layer should be matched to the solution in the bulk flow. The exact relations connecting C_1 and C_2 to σ and m are prohibitively complicated for reproduction here.

4. Large σ limit

Simple analytical results may be obtained in the case of a strong magnetization, $1 \ll \sigma \ll \Gamma^2/4$. The upper limit on σ comes from the fact that for $\sigma \geq \Gamma^2/4$ the shock is no longer strong: it's four-speed become comparable to the upstream Alfvén four-velocity.

Changing to tilde-functions and expanding Eq. (15) for $\sigma \gg 1$ we find

$$
\begin{aligned}
\frac{1}{\tilde g}\frac{\partial \ln \tilde h}{\partial \chi} &= \frac{3\,\tilde h^2\,(m-1)}{2\,(1+m)\left(\tilde f - 3\,\tilde h^2\,\chi\,\tilde g\right)} \\
\frac{1}{\tilde g}\frac{\partial \ln \tilde g}{\partial \chi} &= \frac{3\,\tilde h^2\,(m-1)}{(1+m)\left(\tilde f - 3\,\tilde h^2\,\chi\,\tilde g\right)} \\
\frac{1}{\tilde g}\frac{\partial \ln \tilde n}{\partial \chi} &= \frac{3\,\tilde h^2\,(m-1)}{2\,(1+m)\left(\tilde f - 3\,\tilde h^2\,\chi\,\tilde g\right)}
\end{aligned}
$$

$$\frac{1}{\tilde{g}}\frac{\partial \ln \tilde{f}}{\partial \chi} = \frac{2\,\tilde{h}^2\,(m-1)}{(1+m)\left(\tilde{f} - 3\,\tilde{h}^2\,\chi\,\tilde{g}\right)} \tag{26}$$

Which immediately gives

$$\tilde{g} = \tilde{h}^2, \qquad \tilde{f} = \tilde{h}^{4/3}, \qquad \tilde{n} = \tilde{h} \tag{27}$$

In the strongly magnetized limit the pressure $\tilde{f}$ is proportional to density $\tilde{n}$ to the 4/3 power.

We can then resolve the equations for $\tilde{h}$ as an implicit function of χ:

$$\chi = \frac{2\,(3-m)}{(7-m)}\tilde{h}^{-\frac{2\,(1+m)}{m-1}} + \frac{1+m}{(7-m)}\tilde{h}^{-\frac{8}{3}} \tag{28}$$

(see Fig 3). In particular, for the point explosion case, $m = 3$, this gives

$$\tilde{h} = \chi^{-3/8}, \qquad \tilde{f} = \chi^{-1/2}, \qquad \tilde{g} = \chi^{-3/4}, \qquad \tilde{n} = \chi^{-3/8} \tag{29}$$

The local magnetization parameter $\beta = \tilde{h}^2/\tilde{f} = \chi^{-1/4}$ is a slowly decreasing function of χ.

Relations (29) represent new self-similar solutions for strong point explosion in a strongly magnetized medium with $1 \ll \sigma \ll \Gamma^2/4$.

As a test to these solutions we note (following the arguments of B&M) that for point explosion the energy associated with some interval of $d\chi$ should remain constant. This leads to the condition

$$T^{0r} = T^{00}\beta_N \tag{30}$$

where $\beta_N = \left(\frac{\partial r}{\partial t}\right)_{\chi=const}$. In the leading orders of $1 \ll \sigma \ll \Gamma^2$ this requires

$$\chi = \frac{\tilde{f}}{\tilde{g}\tilde{h}^2} \tag{31}$$

which is satisfied by the relations (29)

The total energy contained inside the relativistic strongly magnetized shock is, for $m = 3$,

$$E \approx 4\pi t^3 \int_0^R \gamma^2 h^2 r^2 dr = \frac{\pi}{4} t^3 n_1 m_i c^3 y \tag{32}$$

which is independent of time. This relation provides the normalization for $y = \Gamma^2$.

For the case of a blast wave with an energy supply we find:

$$E = C(m)\, n_1 m_i c^3 t^3 y^{4/(1+m)} \sim t^{(3-m)/(m+1)} \tag{33}$$

with the constant of proportionality $C(m)$ being a complicated function of m. If the energy source is a power law function of time $L = L_0 t^q$, then

$$m = \frac{2-q}{2+q} \tag{34}$$

(c.f. B&M Eq. (57)).

5. Blast wave in an inhomogeneous medium

In this section we assume the unshocked density and magnetic field have power law dependences on the radius: $n_1 \sim r^{-k}$ and $b_1 \sim r^{-l}$. As a classical example the non-relativistic constant velocity wind gives $k = 2, l = 1$. Straightforward calculations give

$$
\begin{aligned}
&\frac{\mathcal{A}}{g}\frac{\partial \ln f}{\partial \chi} = -2\,f\,\left(4 + m\,\left(\chi g - 4\right) + k\,\left(\chi g - 2\right) - 4\,\chi g\right) + \\
&\frac{h^2\,\left(-2\,\left(l + m - 1\right) + \left(-6 + 3\,k - 2\,l + 3\,m\right)\,\chi g\right)}{\chi g - 1} \\
&\frac{\mathcal{A}}{g}\frac{\partial \ln g}{\partial \chi} = f\,\left(-4 + 3\,k + 7\,m - 2\,\left(2 + m\right)\,\chi g\right) + 3\,h^2\,\left(l + m - 1\right) \\
&\frac{\mathcal{A}}{g}\frac{\partial \ln h}{\partial \chi} = -\frac{f\,\left(-8 + 3\,k + 4\,l + 11\,m + \left(8 + 3\,k - 16\,l - 11\,m\right)\,\chi g + 2\,\left(-4 + 2\,l + m\right)\,\chi^2 g^2\right)}{2\,\left(\chi g - 1\right)} + \\
&\frac{3\,h^2\,\left(l + m - 1\right)}{2} \\
&\frac{\mathcal{A}}{g}\frac{\partial \ln n}{\partial \chi} = \frac{f\,\left(12 - 7\,k - 11\,m + \left(-24 + 13\,k + 11\,m\right)\,\chi g - 2\,\left(-6 + 2\,k + m\right)\,\chi^2 g^2\right)}{2\,\left(\chi g - 1\right)} - \\
&\frac{3\,h^2\,\left(l + m - 1 + \left(3 - 2\,k + l - m\right)\,\chi g\right)}{2\,\left(\chi g - 1\right)} \quad \Big) \\
&\mathcal{A} = \left(1 + m\right)\,\left(2\,f\,\left(1 - 4\,\chi\,g + \chi^2\,g^2\right) - 3\,\chi\,g\,h^2\right)
\end{aligned} \tag{35}
$$

In the limit of small σ we reproduce equations (62-64) of B&M (l naturally falls out) plus an equation for the magnetic field:

$$
\frac{1}{\tilde{g}}\frac{\partial \ln \tilde{h}}{\partial \chi} = -\frac{\left(-16 + 6\,k + 8l + 22\,m + \left(8 + 3\,k - 16\,l - 11\,m\right)\,\chi \tilde{g} + \left(-4 + 2l + m\right)\,\chi^2\,\tilde{g}^2\right)}{2\,\left(1 + m\right)\,\left(\chi\,\tilde{g} - 2\right)\,\left(4 - 8\,\chi\,\tilde{g} + \chi^2\,\tilde{g}^2\right)} \tag{36}
$$

which for the point explosion case $m = 3 - k$ (magnetic energy density is not important) gives

$$
\tilde{g} = \frac{1}{\chi}, \qquad \tilde{h} = \chi^{\frac{k-2\,(4-l)}{2\,(4-k)}}, \qquad \tilde{f} = \chi^{(17-4k)/(3k-12)}, \qquad \tilde{n} = \chi^{(10-3k)/(4(k-4))} \tag{37}
$$

If the upstream medium has a constant magnetization parameter, then $l = k/2$ and $\tilde{h} = 1/\chi$.

In the large σ limit we find $\tilde{g} = h^2$, $\tilde{f} = h^{4/3}$, $\tilde{n} = \tilde{h}$ and an equation for the magnetic field

$$
\frac{1}{\tilde{g}}\frac{\partial \ln \tilde{h}}{\partial \chi} = \frac{3\,\tilde{h}\,\left(l + m - 1\right)}{2\,\left(1 + m\right)\,\left(\tilde{f} - 3\,\tilde{h}^2\,\tilde{g}\,\chi\right)} \tag{38}
$$

which may be resolved for $\chi(h)$:

$$
\chi = \frac{1}{7 - 2l - m}\left((1 + m)\tilde{h}^{-\frac{8}{3}} + 2\,\left(3 - 2l - m\right)\,\tilde{h}^{\frac{2\,(11 - 8\,l - 5\,m)}{3\,(l + m - 1)}}\right) \tag{39}
$$

Again, similarly to the homogeneous case, the point explosion, $m = 3 - 2l$ corresponds to $\tilde{h} = 1/\chi^{3/8}$ The case of constant energy source in a current-free wind ($l = 1$, $m = 0$) gives $\tilde{g} = \tilde{f} = \tilde{h} = \tilde{n} = 1$

Similarly to the homogeneous case we can obtain the solutions inside the magnetosheath for the case of nonzero k and l. We find that the width of the boundary layer in coordinates x is

$$
\Delta x = \sigma^{\frac{12 - 3\,k - m}{2\,(-4 + 3\,k - 4\,l + m)}} \tag{40}
$$

The structure of the magnetic field in the layer is determined implicitly as

$$
x - 1 = C_2\,\left(2 - h^2\right)^{\frac{3\,(-2 + l)}{-4 + 3\,k - 4\,l + m}}\,h^{\frac{-12 + 3\,k + m}{4 - 3\,k + 4\,l - m}} \tag{41}
$$

(compare with (24)). Relation (41) determines the behavior of the magnetic field on the contact discontinuity. For example, for a constant external magnetization parameter $l = k/2$ magnetic field completely dominates the boundary layer dynamics for $k < 4$ and $k + m < 4$. For a shock wave propagating in a non-relativistic constant velocity and constant magnetization wind ($k = 2, l = 1$) magnetic field goes to a constant while density and pressure go to zero for $m < 2$.

6. Discussion

We have analyzed the propagation of a spherically symmetric relativistic shock wave into an inhomogeneous medium permeated by toroidal magnetic field. The main limitation of the current work is the neglect of the possible lateral expansion. The natural and much more complicated extension of this approach would include a blast wave in a constant magnetic field.

The self-similar solutions presented here are generally applicable to the expansion of a strongly relativisticly blast wave in a preexisting spherical cavity blown by the magnetized wind and in a equatorial region of a constant external magnetic field. The self-similar structure of a thin magnetosheath, on the other hand, has a much broader validity since any radial (orthogonal to the surface of the contact discontinuity) magnetic field should approach 0 on the surface of the CD.

The results of this work bear relevance to the physics of Gamma Ray Bursts. In particular, the most popular model of GRBs relies on relativisticly strong shock waves to produce both Gamma-ray and afterglow emission[9]. Previously, the dynamics of the magnetic field has not been included in the model. In addition, the very basic assumptions of the model - presence of near-equipartition magnetic fields and acceleration of particles by relativistic shocks - have been criticized[17]. Our results point to the new interesting possibility that the relativistic flow may produce radiation effectively in the magnetosheath layer adjacent to the contact discontinuity between two flows where magnetic field reaches near-equipartition values. We leave the investigation of this possibility to a future work.

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Appendix A: Dynamic equations for independent variable $x = g\chi$

Changing to the new variable $x = g\chi$ in (15) and using

$$dx = \frac{\left(-3\,h^2\,(-2+l)\,x + f\,\left(-2+m\,(x-2) - 3\,(k-4)\,x + 2\,x^2\right)\right)\,g(x)}{(1+m)\,\left(3\,h^2\,x - 2\,f\,(1-4\,x+x^2)\right)}\,d\chi \tag{A1}$$

we find

$$\begin{aligned}
(x-1)\,\mathcal{A}'\,\frac{\partial \ln f}{\partial x} &= 2\,f\,\left(4+m\,(-4+x)+k\,(x-2)-4\,x\right)\,(x-1) \\
&\quad + h^2\,\left(2\,(l+m-1)+(6-3\,k+2\,l-3\,m)\,x\right) \\
\mathcal{A}'\,\frac{\partial \ln g}{\partial x} &= f\,\left(4-3\,k-7\,m+2\,(2+m)\,x\right) - 3\,h^2\,(l+m-1) \\
(x-1)\,\mathcal{A}'\,\frac{\partial \ln h}{\partial x} &= \frac{1}{2} f\,\left(-8+3\,k+4\,l+11\,m+(8+3\,k-16\,l-11\,m)\,x+2\,(-4+2\,l+m)\,x^2\right) - \\
&\quad \frac{3}{2} h^2\,(l+m-1)\,(x-1) \\
(x-1)\,\mathcal{A}'\,\frac{\partial \ln n}{\partial x} &= \frac{1}{2}\left(-12+7\,k+11\,m+(24-13\,k-11\,m)\,x+2\,(-6+2\,k+m)\,x^2\right) f + \\
&\quad \frac{3}{2}\left(l+m-1+(3-2\,k+l-m)\,x\right) h^2 \\
\mathcal{A}' &= -3\,h^2\,(-2+l)\,x + f\,\left(-2+m\,(x-2)-3\,(k-4)\,x+2\,x^2\right)
\end{aligned} \tag{A2}$$

In coordinate x the shock is located at $x_0 = g_0$ and the contact discontinuity is located at $x = 1$.

Appendix B: Lower dimension outflows

Without giving detail calculations, we comment here on the generalization of the above results for the lower dimensional systems, when the outflow is cylindrically or linearly symmetric. We give here only the scalings for the point explosion cases in a homogeneous medium, generalizations for inhomogeneous media with energy supply are straightforward. For 1-D case (strong shock propagating along a tube) the solutions look very similar to the 3-D case. In the small σ limit we find $\tilde{g} = \tilde{h} = \tilde{n} = 1/\chi$, $\tilde{f} = \chi^{-7/6}$, while for the large σ limit we reproduce relations (29). The structure of the equations in the 2-D case is, in fact, *qualitatively* different from the 1-D and 3-D cases. In the small σ limit: $\tilde{g} = 1/\chi$, $\tilde{h} = 1/\chi^{5/6}$, $\tilde{n} = 1/\chi^{7/6}$, $\tilde{f} = \chi^{-4/3}$, while the large σ turns out to be unusual: $\tilde{g} = \chi^{-5/6}$, $\tilde{n} = \tilde{h} = \chi^{-5/12}$, $\tilde{f} = \chi^{-2/3}$.

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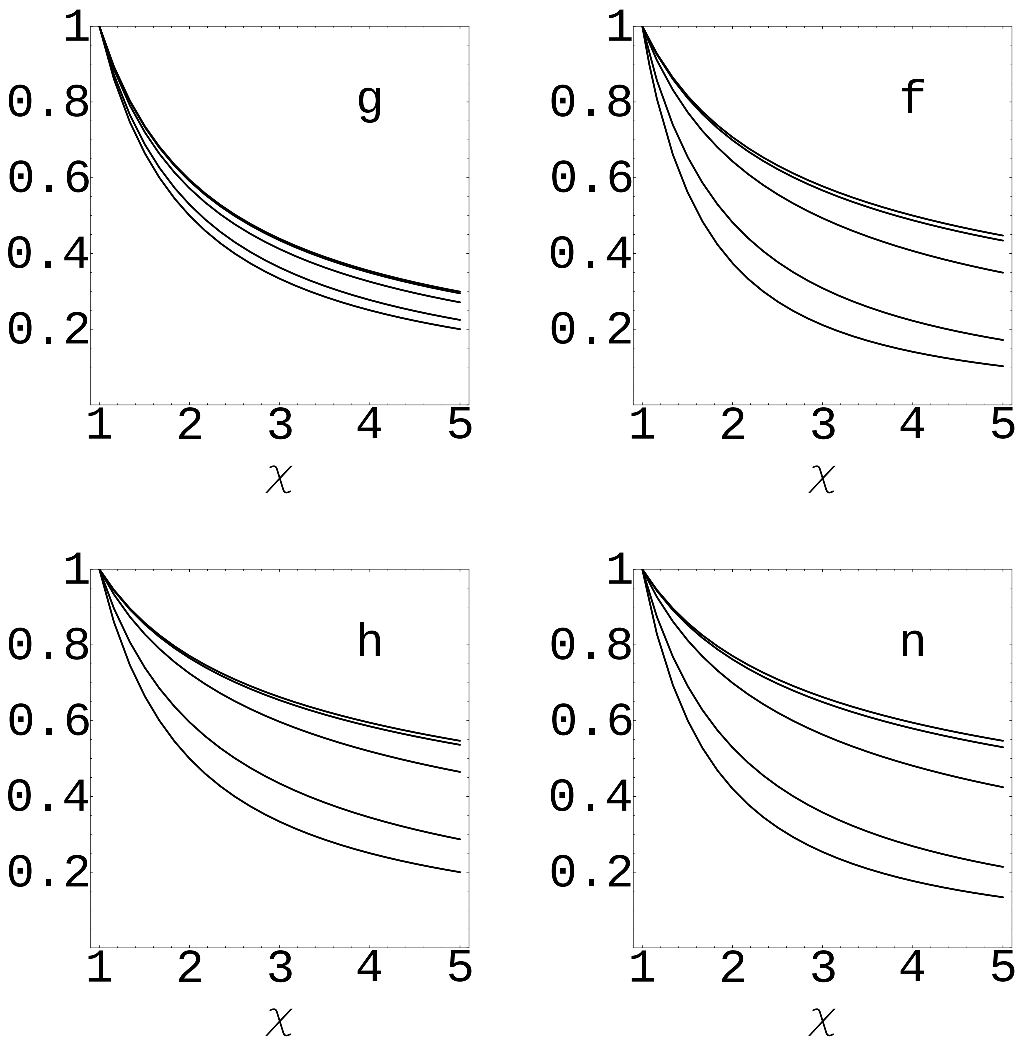

Fig. 1. Lorentz factors, pressure, magnetic field and density for $m = 3$. The curves from top to bottom are (i) asymptotic limit $\sigma \gg 1$, (ii) $\sigma = 10$, (iii) $\sigma = 1$, (iv) $\sigma = 0.1$ and $\sigma = 0$. In the case $\sigma = 0$ the magnetic field is normalized to some arbitrarily small initial value.

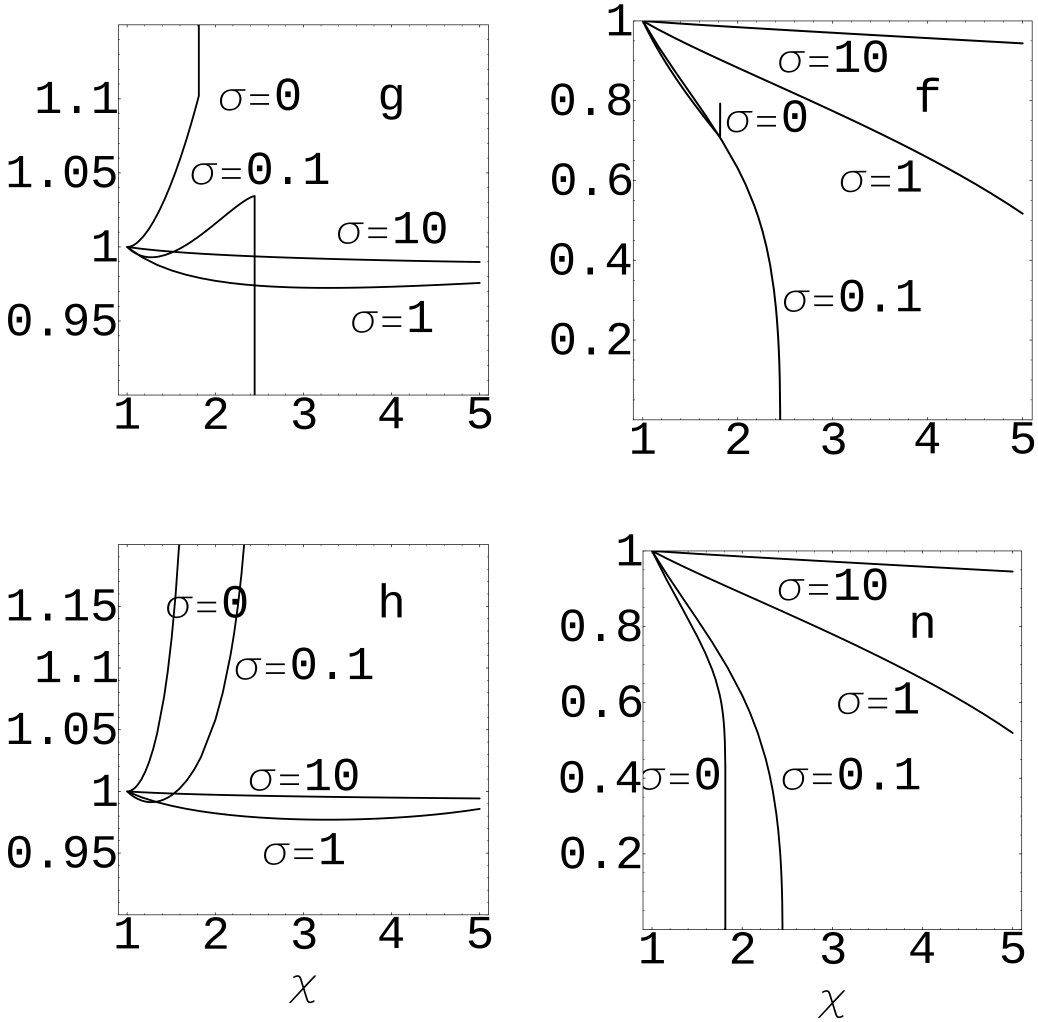

Fig. 2. Lorentz factors, pressure, magnetic field and density for $m = 1$. The curves are labeled by the values of σ. For $\sigma \gg 1$ the asymptotic solutions are $g = f = h = n = 1$. Note that magnetic field piles up on the contact discontinuity.

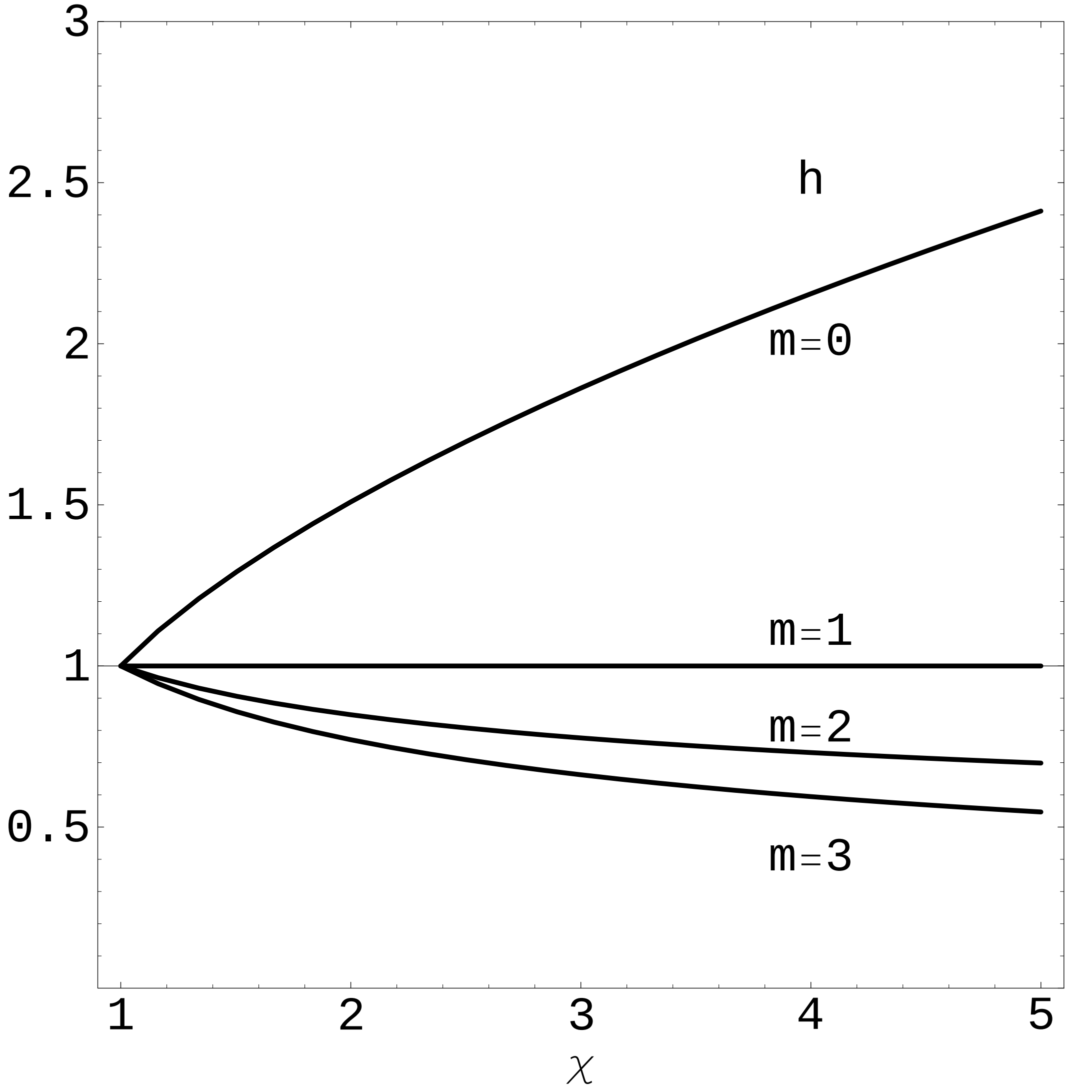

Fig. 3. Magnetic field $\tilde{h}$ in the strongly magnetized limit $\sigma \gg 1$ for different values of m ($k = l = 0$). Other functions are powers of $\tilde{h}$ (eq. 27).